A FORMAL SOLUTION OF CERTAIN SIMULTANEOUS QUADRUPLE INTEGRAL EQUATIONS INVOLVING I-FUNCTIONS

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ABSTRACT

The problem discussed is to obtain the solution of simultaneous quadruple integral equations involving I-functions. The method followed is that of fractional integration. The given simultaneous quadruple integral equations have been transformed by the application of fractional Erdelyi-Kober operators to four others simultaneous integral equations with a common Kernel. Here for the sake of generality the I-function is assumed as unsymmetrical Fourier kernel. Here with the help of theorems of Mellin transform, the solution of simultaneous Quadruple Integral equations is obtained. Some interesting particular cases have been derived.

KEY WORDS : 45-XX Integral Equations, 45F10 Quadruple Integral Equations, 33C60 Saxena's I – function, 45 H05 (Special kernels) unsymmetrical Fourier kernel, 33C XX Hypergeometric Function, 44A15 Special Transforms (Mellin), 26A33 Fractional derivatives and Integrals, 44A20 Transforms of special functions, 45E99 Suitable contour Barnes type.

INTRODUCTION

V.P. Saxena [14] defined the I-function, which is more general hypergeometric function than the Fox's H-Function. Saxena's I-function is defined as,

$$I^{m,n}_{p_i;q_i;r}[z] = I^{m,n}_{p_i;q_i;r}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n}, & (a_{j_i},\alpha_{j_i})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}, & (b_{j_i},\beta_{j_i})_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi i}\int_L t(s) z^s\, ds \qquad [1.1]$$

Where

$$t(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j-m+1}^{q_i}\Gamma(1-b_{j_i}+\beta_{j_i} s)\prod_{j-n+1}^{p_i}\Gamma(a_{j_i}-\alpha_{j_i} s)\right\}} \qquad [1.2]$$

Here p_i and q_i are positive integers and m, n are integers satisfying

$0 \le n \le p_i$, $0 \le m \le q_i$, (i=1, 2,…………………..,r), r is finite-

$\alpha_j, \beta_j, \alpha_{j_i}, \beta_{j_i}$ are real and positive and $a_j, b_j, a_{j_i}, b_{j_i}$ are complex numbers such that

$\alpha_1 j\,(b_{1n}+\nu) \neq \beta_1 h\,(\alpha_1 j-1-\nu)$ for ν= 0,1,2,…………… ; h=1,2,……………m ; j=1,2,……………,r.

L is a suitable Contour of Barnes type which runs from σ-i∞ to σ+i∞, (σ is real) in the complex s-plane such that the points

$s = (\alpha_j\text{-}1\text{-}\nu) / \alpha_j$: j=1,2,……………,n ; ν= 0,1,2,…………… and

$s=(b_j+\nu) / \beta_j$: j=1,2,……………m; ν= 0,1,2,…………… lie to the left hand side and right hand side of the contour L respectively. The conditions under which [7.1.1] converges are given as follows:

$$A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_j \quad : \tag{1.3}$$

(i=1,2,…, …..,r)

$$B_i(i) = 1/2(p_i(i) - q_i(i)) + \sum_{j=1}^{q_i} b_j - \sum_{j=1}^{n} a_j \quad : \tag{1.4}$$

(i=1,2,……..,r)

$$A_i > 0 , |\arg z| < \frac{1}{2}\pi A_i \text{ and } B_i \geq 0 \qquad : (i=1,2, ………..,r) \tag{1.5}$$

Mathur [155] obtained the solution of simultaneous dual integral equations involving I-functions.

Mathur [156] have also considered the formal solution of triple integral equations involving I-functions. The aim of the present section is to obtain the solution of the quadruple integral equations involving I-functions. The method followed is that of fractional integral operators. By the application of fractional integral operators given equations are transformed into a equation with common Kernel.

RESULTS USED IN THE PROOF OF THE SEQUEL

MELLIN TRANSFORM:

$$M\{f(x)\} = F(s) = \int_0^\infty f(x) x^{s-1} dx \tag{2.1}$$

INVERSE MELLIN TRANSFORM C+I∞ :

$$M^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s) x^{-s} ds \tag{2.2}$$

For s = c+in, x>0

PERSAVAL THEOREM FOR MELLIN TRANSFORMS:

Let M {f (u)} = F(s) and M {a (u)} = A(s) then

M {a (ux)} = $x^{-s}A(s)$ and

$$\int_0^{\infty} f(ux)a(u)du = \frac{1}{2\pi i}\int_L x^{-s}F(s)A(1-s)ds$$ [2.3]

FRACTIONAL INTEGRAL FORMULAE:

Fractional Integral formulae have been defined by Fox as follows:

$$\int_0^{x} \left[\!\left[(x)\right]\!\right]^{\frac{1}{c}} - v^{\frac{1}{c}}\Big)^{d-e-1} . v^{\frac{e}{c}-s-1}dv = \frac{e\,\Gamma(d-e)\Gamma(e-cs)}{\Gamma(d-cs)} . x^{\frac{d}{c}-\frac{1}{c}-s}$$ [2.4]

provided d $> e$ *and* $\frac{e}{c} > \sigma$ *where* $s - \sigma + it$ *and* $0 < x < 1$

and

$$\int_x^{\infty} \left[\!\left[(v)\right]\!\right]^{\frac{1}{c}} - x^{\frac{1}{c}}\Big)^{d-e-1} . v^{\frac{1}{c}-\frac{d}{c}-s-1}dv = \frac{c\,\Gamma(d-e)\Gamma(e+cs)}{\Gamma(d+cs)} . x^{-\frac{e}{c}-s}$$ [2.5]

provided d $> e$ *and* $\frac{e}{c} > \sigma$ *where* $s = \sigma + it$ *and* $x > 1$

FRACTIONAL ERDELYI – KOBER OPERATORS:

Fox used the following generalized Erdelyi –Kober Operators:

$$T\,[\gamma, \epsilon : m]\{f(x)\} - \frac{m}{\Gamma\gamma} x^{-\gamma m-\epsilon+m-1} \int_0^{x} \left[\!\left[(x)\right]\!\right]^{m} - v^{m})^{\gamma-1} . v^{\epsilon} f(v)dv,$$ [2.6]

where $0 < x < 1$

and

$$R\,[\gamma, \epsilon : m]\{f(x)\} = \frac{m}{\Gamma\gamma} . x^{\epsilon} \int_x^{\infty} \left[\!\left[(v)\right]\!\right]^{m} - x^{m})^{\gamma-1} . v^{-\epsilon-\gamma m+m-1} f(v)dv,$$ [2.7]

where $x > 1$

The operator T exists if f(x) $CL_p\,(0,\infty), p > 1, \gamma > 0$ *and* $\epsilon > \frac{1-p}{p}$ **and if f(x) can be differentiated** sufficient number of times then the operator T exists for both negative and positive value of γ.

The operator R exists if $f(x) \in L_p(0,\infty), p \geq 1$ and If $f(x)$ can be differentiated sufficient number of times then the operator R exists.

If m > $\epsilon > \frac{-1}{p}$ while γ can take any negative or positive value.

A THEOREM FOR MELLIN TRANSFORMS:

If M {f (u)} = F(s) and M {g (u)} = G (s) then

$$\int_0^\infty g(u) f(u)\, du = \frac{1}{2\pi i} \lim_{\substack{T\to\infty \\ \&\sigma_0 = R_e(s)}} \int_{\sigma_0 - iT}^{\sigma_0 + iT} G(s) F(1-s)\, ds \qquad [2.8]$$

Thus if f (ux) is considered to be a function of u with x as a parameter, where x > 0 then

$$M\{f(ux)\} = x^{-s} M\{f(u)\} \qquad [2.9]$$

From [2.8] and [2.9] we have

$$\int_0^\infty g(ux) f(u)\, du = \frac{1}{2\pi i} \lim_{\substack{T\to\infty \\ \&\sigma_0 = R_e(s)}} \int_{\sigma_0 - iT}^{\sigma_0 + iT} x^{-s} G(s) F(1-s)\, ds \qquad [2.10]$$

Additional conditions for the validity of [2.10] are that

F(s) $\in L_p(\sigma_0 - i\infty, \sigma_0 + i\infty)$ and

$x^{1-\sigma_0} g(x) \in L_p(0,\infty), p \geq 1$ where L_p denotes the class of functions g(x) such that

$$\int_0^\infty |g(x)|^p \cdot \frac{dx}{x} < \infty \qquad [2.11]$$

3. THE SOLUTION OF FOLLOWING QUADRUPLE INTEGRAL EQUATIONS INVOLVING I-FUNCTIONS

∫↓0↑∞▒〖 I↓(p↓(i); 〖 q〗↓(i);r)↑(m, n) [ux/▪((〖a▀(k@j), 〗↓ α↓j)↓(1,n

[3.1]

where $x \in (0, a)$

∫↓0↑∞▒〖 I↓(p↓(i); 〖 q〗↓(i);r)↑(m, n) [ux/▪((〖c▀(k@j), 〗↓ α↓j)↓(1,n),@(〖b▪(k@j), 〗↓ β↓j)↓(1,m)

where $x \in (a, b)$

∫↓0↑∞▒ 〖 I↓(p↓(i): 〖 q 〗↓(i); r)↑(m, n) [ux/■((〖c█(k@j), 〗↓ α↓j)↓(1, n), ¢

[3.3]

where x∈(b, c)

∫↓0↑∞▒ 〖 I↓(p↓(i): 〖 q 〗↓(i); r)↑(m, n) [ux/■((〖c█(k@j), 〗↓ α↓j)↓(1, n), ¢

[3.4]

where x∈(c, ∞)

Where $0 < a < 1$ and $1 < c < \infty$.

Here a_{hk}, b_{hk}, c_{hk} and d_{hk} are well known constants and $\phi_{1_k}(x)$, $\phi_{2_k}(x)$, $\phi_{3_k}(x)$ and $\phi_{4_k}($
are prescribed functions for h = 1,2,……..,n ; k = 1,2,………,n and $f_h(x)$ is unknown function for h = 1,2,………n, which is to be determined. Applying Persaval’s theorem in the integral equations [3.1], [3.2] ,[3.3] and [3.4] under the conditions:

■(−Min@1 ≤ j ≤ m) 〖R↓(e) (〗 〖b■(k@j)■(@)〗↓ /βj) < Re (s) < 1/□ αj -
■(−Max@1 < j < n)┬ 〖R↓(e) (〗 〖a■(k@j)〗↓(■(@)) /αj) [3.5]

■(−Min@1 ≤ j ≤ m) 〖R↓(e) (〗 〖b■(k@j)〗↓ /βj) < Re (s) < 1/□ αj -
■(−Max@1 ≤ j ≤ n) 〖R↓(e) (〗 〖c■(k@j)〗↓(■(@)) /αj) [3.6]

■(−Min@1 ≤ j ≤ m) 〖R↓(e) (〗 〖d■(k@j)〗↓ /βj)< Re (s) < 1/□ αj -
■(−Max@1 ≤ j ≤ n) 〖R↓(e) (〗 〖c■(k@j)〗↓ /αj) [3.7]

$$|\arg x| < \frac{1}{2}\pi D_i \quad , \qquad i = 1,2,\ldots\ldots\ldots\ldots,r. \tag{3.8}$$

Where

$$D_i = \sum_{j=1}^{m} \beta_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=m+1}^{q_i} \beta_j - \sum_{j=1}^{m} \alpha_j \tag{3.9}$$

where i = 1, 2, ..., r.

then the integral equations [3.1], [3.2] ,[3.3] and [3.4] are reduced to the following forms:

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}^{k}+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_{j}^{k}-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{j_i}^{k}-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{j_i}^{k}+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} a_{hk}\, F_h(1-s)ds = \phi_{1_k}(x) \qquad [3.10]$$

where $x \in (0,a)$.

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}^{k}+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_{j}^{k}-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{j_i}^{k}-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{j_i}^{k}+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} b_{hk}\, F_h(1-s)ds = \phi_{2_k}(x). \qquad [3.11]$$

where $x \in (a,b)$.

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j-1}^{m}\Gamma\left(b_{j}^{k}+\beta_j s\right)\prod_{j-1}^{n}\Gamma\left(1-c_{j}^{k}-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{j_i}^{k}-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{j_i}^{k}+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} c_{hk}\, F_h(1-s)ds = \phi 3_k(x) \qquad [3.12]$$

.

where $x \in (b,c)$.

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(d_j^k + \beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1 - c_j^k - \alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{j_i}^k - \beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{j_i}^k + \alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} d_{hk} F_h(1-s)ds = \phi_4(x) \quad [3.13]$$

where $x \in (c, \infty)$.

Here M {f**h** (u)} = F**h** (s) for h = 1,2,……………,n. [3.14]

Now in integral equation [3.10] replacing x by ν and multiplying both sides of the equation [3.10] by

$\left((x)^{\frac{1}{\alpha_n}} - \nu^{\frac{1}{\alpha_n}}\right)^{c_n^k - a_n^k - 1} . \nu^{\frac{1-c_n^k}{\alpha_n}-1}$ and integrating both sides of integral equation [3.10]with respect to ν from 0 to x where x ϵ (0,a) with 0 < a < 1 and applying well known fractional integral formula [2.4] in equation [3.10], we find

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j^k + \beta_j s\right)\prod_{j=1}^{n-1}\Gamma\left(1 - a_j^k - \alpha_j s\right)\Gamma\left(1 - c_n^k - \alpha_n s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{j_i}^k - \beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{j_i}^k + \alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} a_{hk} F_h(1-s)ds =$$

$$\frac{1}{\alpha_n \Gamma\left(c_n^k - a_n^k\right)} x^{a_n^k/\alpha_n}\int_0^x \left((x)^{\frac{1}{\alpha_n}} - \nu^{\frac{1}{\alpha_n}}\right)^{c_n^k - a_n^k - 1} . \nu^{\frac{1-c_n^k}{\alpha_n}-1}\phi_{1_k}(\nu)d\nu \quad [3.15]$$

where $0 < x < 1$.

Using the Erdelyi-Kober operator T from [2.6] in equation [3.15],

For brevity we write,

$$T\left[c^{k}_{j} - a^{k}_{j}, \; (1- [c_{(k@j)}]_{i})/\alpha_{i}j - 1 : 1/\alpha_{i}j\right] \{\phi_{i}(1_{(@k)})(x)\} = T_{i}j\{\phi_{i}(1_{(@k)})(x)\} \quad [3.16]$$

where $x \in (0, a)$.

then

$$T\left[c^{k}_{n} - a^{k}_{n}, \frac{1-c^{k}_{n}}{\alpha_n} - 1 : \frac{1}{\alpha_n}\right]\{\phi_{1_k}(x)\} = T_n\{\phi_{1_k}(x)\} \quad [3.17]$$

where $x \in (0, a)$.

Hence from [3.17], the integral equation [3.15] can be written as

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b^{k}_{j} + \beta_j s\right)\prod_{j=1}^{n-1}\Gamma\left(1 - a^{k}_{j} - \alpha_j s\right)\Gamma\left(1 - c^{k}_{n} - \alpha_n s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1 - b^{k}_{j_i} - \beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a^{k}_{j_i} + \alpha_{j_i} s\right)\right\}}$$

$$x^{s}\sum_{h=1}^{n} a_{hk}\, F_h(1-s)\,ds = T_n\{\phi_{1_k}(x)\}. \quad [3.18]$$

where $x \in (0, a)$.

Now repeating the same process in integral equation [3.18] for j= n-1,

n-2,…….,3,2,1then the integral equation [3.18] takes the from

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j^k+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_j^k-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}^k-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^k+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} a_{hk}F_h(1-s)\,ds=\prod_{j=1}^{n}T_j\left\{\phi_{1_k}(x)\right\}. \qquad [3.19]$$

where $x \in (0,a)$, k= 1,2,…………..,n

or

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j^k+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_j^k-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}^k-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^k+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} b_{hk}F_h(1-s)\,ds=\sum_{h=1}^{n} e_{hk}\prod_{j=1}^{n}T_j\left\{\phi_{1_k}(x)\right\}, \qquad [3.20]$$

where $x \subset (0,a)$

, k= 1,2,………………,n and e_{hk} are the elements of the matrix

$$[b_{hk}][a_{hk}]^{-1}$$

We have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j^k+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_j^k-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}^k-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^k+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} c_{hk} F_h(1-s)ds = \phi 3_k(x) \qquad [3.12]$$

where $x \in (b, c)$.

or

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j^k + \beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1 - c_j^k - \alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{ji}^k - \beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^k + \alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} b_{hk} F_h(1-s)ds = \sum_{h=1}^{n} f_{hk}\ \phi 3_k(x) \text{ where } x \in (b, c) \qquad [3.21]$$

.

k= 1,2,……………,n and f_{hk} are the elements of the matrix

$$[b_{hk}][c_{hk}]^{-1}$$

Now in integral equation [3.13] replacing x by ν and multiplying both sides of the equation [3.13] by

$$\left((\nu)^{\frac{1}{\beta_m}} - x^{\frac{1}{\beta_m}}\right)^{d_m^k - b_m^k - 1} . \nu^{\frac{1-d_m^k}{\beta_m} - 1}$$

and integrating both sides of integral equation [3.13] with respect to ν from x to ∞ , Where $x \in (c, \infty)$ with c > 1 and applying well known fractional integral formula [2.5], We find,

$1/2\pi i \int_L \left(\prod_{j=1}^{m}\Gamma(d_j^k + \beta_j s)\prod_{j=1}^{n}\Gamma(1 - c_j^k - \alpha_j s)\,\Gamma(b_m^k + \beta_m s)\right)/\left(\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\right.\right.$

$$= \frac{1}{\beta_m \Gamma\left(d_m^k - b_m^k\right)} . x^{\frac{b_m^k}{\beta_m}} \int_x^{\infty} \left((\nu)^{\frac{1}{\beta_m}} - x^{\frac{1}{\beta_m}}\right)^{d_m^k - b_m^k - 1}$$

$$(\nu)^{\left(1 - d_m^k\right)/\beta_m - 1}\ \phi_4^k(\nu)\, d\nu \qquad [3.22]$$

where x > 1.

Using the Erdely-Kober operator R from [2.7] in equation [3.22]

For breviety we write,

$$R\left[d_j^k - b_j^k, \frac{b_j^k}{\beta_j} : 1/\beta_j\right] \{\phi_4^k(x)\} = R_j \{\phi_4^k(x)\} \qquad [3.23]$$

where $x \in (c, \infty)$.

then

$$R\left[d_m^k - b_m^k, \frac{b_m^k}{\beta_m} : 1/\beta_1 m\right] \{\phi_1(4(@k))(x)\} = R_1 m \{\phi_1(4(@k))(x)\}$$ [3.24]

wherex $\in (c, \infty)$.

Hence from [3.24], the integral equation [3.22] can be written as

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m-1} \Gamma\left(d_j^k + \beta_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - c_j^k - \alpha_j s\right) \Gamma\left(b_m^k + \beta_m s\right)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji}^k - \beta_{j_i} s\right) \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji}^k + \alpha_{j_i} s\right) \right\}}$$

$$x^{-s} \sum_{h=1}^{n} d_{hk} F_h(1 - s) ds = R_m\{\phi_{4_k}(x)\}$$ [3.25]

wherex $\in (c, \infty)$.

Now repeating the same process in integral equation [3.25] for

j=m-1,m-2,………….,3,2,1 then the integral equation [3.25] takes the form

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma\left(b_j^k + \beta_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - c_j^k - \alpha_j s\right)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji}^k - \beta_{j_i} s\right) \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji}^k + \alpha_{j_i} s\right) \right\}}$$

$$x^{-s} \sum_{h=1}^{n} d_{hk} F_h(1 - s) ds = \prod_{j=1}^{m} R_j\{\phi_{4_k}(x)\}.$$ [3.26]

where $x \in (c, \infty), k = 1, 2, \ldots\ldots\ldots\ldots, n.$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}^{k}+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_{j}^{k}-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}^{k}-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^{k}+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} b_{hk}F_h(1-s)ds=\sum_{h=1}^{n} g_{hk}\prod_{j=1}^{m} R_j\left\{\phi_{4_k}(x)\right\} \text{ where } x\in(c,\infty) \quad [3.27]$$

$,k=1,2,\ldots\ldots\ldots\ldots\ldots\ldots,n$ and g_{hk} the elements of the matrix

$$[b_{hk}][d_{hk}]^{-1}$$

Now if we set

P(x) =

$$\left\{\sum_{h=1}^{n} e_{hk} \prod_{j=1}^{n} T_j\left\{\phi_{1k}(x)\right\}\right\}, \quad \text{where}$$

[3.28]

Where k= 1,2,……………………….,n.

then integral equations [3.20],[3.11] ,[3.21] and [3.27] having common kernel can be put into the compact form as

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_{j}^{k}+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-c_{j}^{k}-\alpha_j s\right)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}^{k}-\beta_{j_i} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}^{k}+\alpha_{j_i} s\right)\right\}}$$

$$x^{-s}\sum_{h=1}^{n} b_{hk}F_h(1-s)ds=p_k(x). \quad [3.29]$$

where $x \in (0,\infty)$ and k= 1,2,………………..,n.

In order to solve the integral equation [3.26] , the additional condition required is that the I-function is symmetrical or unsymmetrical Fourier kernel and f(u) is continuous at u=x. For the sake of generality we assume that I-function is un symmetrical Fourier kernel, In this case the parameters have to satisfy the following set of conditions:

(i) $$\sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_j = \sum_{j=m+1}^{q_i}\beta_j - \sum_{j=1}^{n}\alpha_j$$: (i=1,2,……..,r)

(ii) $$\sum_{j=1}^{m} b_j^k - \sum_{j=n+1}^{p_i} a_j^k = \sum_{j=m+1}^{q_i} b_j^k - \sum_{j=1}^{n} c_j^k$$: (i=1, 2, …..,r) [3.30]

(iii) $\mathrm{Re}[(1 - c_j^{(k)}) - \alpha_j/2] > \alpha_j/(2D_i)$; (j=1,2,……..,n ; i=1,2,…..r)

(iv) $\mathrm{Re}[(a_j^{(k)}) + \alpha_j/2] > \alpha_j/(2D_i)$; (j=n+1,2,… …., p_i ; i=1,2,…..r)

[3.31]

(v) $\mathrm{Re}[(d_j^{(k)}) + \beta_j/2] > \beta_j/(2D_i)$; (j=1,2,….,m ; i=1,2,…..r)

(vi) $\mathrm{Re}[(1 - b_j^{(k)}) - \beta_j/2] > \beta_j/(2D_i)$; (j=m+1,…., q_i ; i=1,2,…..r)

Where

$$D_i = \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_j + \sum_{j=m+1}^{q_i}\beta_j - \sum_{j=1}^{n}\alpha_j$$ [3.32]

where i=1,2,…..,r.

Now with the help of theorem V. P Saxena [15], we find its respective reciprocal kernel

$m+1,\ q_i$

[3.33]

Where k= 1,2,…………………..,n.

Taking Mellin transform and making use of the theorem of V. P. Saxena (1971) in integral equation [3.29], We have the following solution:

$$f_h(x) =$$

$$\frac{1}{2\pi i}\left[\lim \sum_{h=1}^{n} t_{hk}\right] \int_{T\to\infty} \int_{1/2-iT}^{1/2+iT} \left(\prod_{i=1}^{r} \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_j^k - \beta_j + \beta_j s\right) \prod_{j=n+1}^{p_i} \Gamma\left(c_j^k \ldots\right.\right. \qquad [3.34]$$

$$. x^{-s} P_k(1-s)ds$$

where M {pk (x)} = Pk (s), thk are the elements of the matrix $[b_{hk}]^{-1}$, h= 1,2,……….,n and k= 1,2,……………,n.

Now using Parseval's theorem in [3.34], we find the solution finally,

$$f_h(x) = \sum_{h=1}^{n} t_{hk} \int_0^{\infty} \sum_{i=1}^{r} H_i[ux] P_k(u)du \qquad [3.35]$$

4. PARTICULAR CASE

If we put r=1 then the integral equations [3.1], [3.2] ,[3.3] and [3.4] reduce to the Quadruple Integral equations involving Fox's H-functions as given below:

$$\int_0^{\infty} H^{m,n}_{p+n,q+m}\left[ux \,\middle|\, \begin{matrix} (a_j^k, \alpha_j) \\ (b_j^k, \beta_j) \end{matrix}\right] \sum_{h=1}^{n} a_{hk} f_h(u)du = \phi_{1k}(x) \qquad [4.1]$$

where $x \in (0, a)$

$$\int_0^{\infty} H^{m,n}_{p+n,q+m}\left[ux \,\middle|\, \begin{matrix} (c_j^k, \alpha_j) \\ (b_j^k, \beta_j) \end{matrix}\right] \sum_{h=1}^{n} b_{hk} f_h(u)du = \phi_{2k}(x) \qquad [4.2]$$

where $x \in (a, b)$.

$$\int_0^\infty H^{m,n}_{p+n,\,q+m}\left[ux \,\middle|\, \begin{matrix} (c^k_j,\alpha_j) \\ (b^k_j,\beta_j) \end{matrix}\right] \sum_{h=1}^{n} c_{hk}\, f_h(u)\,du = \phi_{3k}(x) \qquad [4.3]$$

where $x \in (b, c)$.

$$\int_0^\infty H^{m,n}_{p+n,\,q+m}\left[ux \,\middle|\, \begin{matrix} (c^k_j,\alpha_j) \\ (d^k_j,\beta_j) \end{matrix}\right] \sum_{h=1}^{n} d_{hk}\, f_h(u)\,du = \phi_{4k}(x) \qquad [4.4]$$

where $x \in (c, \infty)$

Where $0<a<1$ and $1<c<\infty$. Here a_{hk}, b_{hk}, c_{hk} and d_{hk} are well known constants and $\phi_{1k}(x)$, $\phi_{2k}(x)$, $\phi_{3k}(x)$ and $\phi_{4k}(x)$ are prescribed functions for

$h= 1,2,\ldots\ldots,n$; $k= 1,2,\ldots\ldots,n$.

The solution of Quadruple Integral equations [4.1], [4.2], [4.3] and [4.4] involving Fox's H-functions is obtained by putting r =1 in solution [3.35] and so ,far solution obtained is,

$$f_h(x) = \sum_{h=1}^{n} t_{hk} \int_0^\infty H[ux] P_k(u)\,du \qquad [4.5]$$

Where $M[p_k(x)] = P_k(s)$ and t_{hk} are the elements of the matrix $[b_{hk}]^{-1}$ for

$h= 1,2,\ldots\ldots,n$; $k= 1,2,\ldots\ldots,n$.

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