QUADRUPLE INTEGRAL EQUATIONS INVOLVING FOX’S H-FUNCTIONS

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ABSTRACT

In the present paper the formal solution of certain quadruple integral equations involving Fox’s H-functions of order n is obtained by the method of fractional integration which is extension of dual integral equations considered by Fox [5]. Here by the application of fractional integration operators, the given quadruple equations are transformed into other quadruple equations with a common kernel and the problem then reduced to that of solving one integral equation.

KEYWORDS: 45F10 Quadruple integral equations, 45H05 Special Kernels (Unsymmetric Fourier Kernel), 44A15 Special Transforms (Mellin Transform), 33C60 Fox’s function, 33CXX Hypergeometric functions, 26A33 Fractional derivatives and integrals, 44A20 Transforms of special functions, 45E99 Suitable contour Barnes type.

INTRODUCTION

Fox,C. [1965] defined the H- function of order n as follows:

$$H\left(x\middle|\begin{matrix}\alpha_i, & a_i\\ \beta_i, & a_i\end{matrix}:n\right) = H\left(x\middle|\begin{matrix}\alpha_1, & a_1\\ \beta_1, & a_1\end{matrix}:\begin{matrix}\alpha_2, & a_2\\ \beta_2, & a_2\end{matrix}:\ldots\ldots:\begin{matrix}\alpha_n, & a_n\\ \beta_n, & a_n\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)}\right\} x^{-s}\, ds \qquad [1.0]$$

We assume that the following conditions are satisfied; some of these conditions are necessary and some serve to simplify the problem:

(i). a_i , α_i , β_i are all real , i= 1,2,………,n.

(ii). $a_i > 0$, $i = 1,2,\ldots\ldots,n$.

(iii). Let s = $\sigma + it$, where σ and t are real ; then the contour C along which the Integral [1.0] is taken, is the straight line whose equation is $\sigma = \sigma_0$ where σ_0 is a constant. This line is parallel to the imaginary axis in the complex s Plane.

(iv). All the poles of the integrand of [1.0] are simple and lie to the left of the Line $\sigma = \sigma_0$. This requires $\sigma_0 > -\alpha_i/a_i$, i = 1,2,……….,n.

Together with these conditions we require either (va) or (vb) below :

(va). $2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n}(\beta_i - \alpha_i)$,

(vb). $2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n}(\beta_i - \alpha_i) - 1$.

The integral [1.0] , taken along the line $\upsilon = \upsilon_0$, converges if condition (va) holds and converges absolutely if condition (vb) holds. This can be verified by using the asymptotic expansion of the Gamma function.

The asymptotic expansion also shows that the contour C of [1.0] can be closed by a large semicircle on the left. On computing the residues we then find that the H function of order n can be expressed as the sum of n power series , the i^{th} of which is multiplied by x^{α_i/a_i} , Each of these power series is an entire function.

The quadruple integral equations we consider here are as follows :

$$\int_0^\infty H\left(ux \,\middle|\, \begin{matrix}\alpha_i & a_i \\ \beta_i & a_i\end{matrix} : n\right) f(u)\,du = \psi_1(x), \qquad x \in (0,a) \qquad [1.1]$$

$$\int_0^\infty H\left(ux \,\middle|\, \begin{matrix}\alpha_i & a_i \\ \mu_i & a_i\end{matrix} : n\right) f(u)\,du = \psi_2(x), \qquad x \in (a,b) \qquad [1.2]$$

$$\int_0^\infty H\left(ux \,\middle|\, \begin{matrix}\alpha_i & a_i \\ \mu_i & a_i\end{matrix} : n\right) f(u)\,du = \psi_3(x), \qquad x \in (b,c) \qquad [1.3]$$

$$\int_0^\infty H\left(ux \,\middle|\, \begin{matrix}\lambda_i & a_i \\ \mu_i & a_i\end{matrix} : n\right) f(u)\,du = \psi_4(x), \qquad x \in (c,\infty) \qquad [1.4]$$

Where $0 < a < 1$ and $1 < c < \infty$ and $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$ and $\psi_4(x)$ are prescribed functions and f(x) is to be determined here.

the constants a_i, i=1,2,……………,n, are the same for the equations [1.2], [1.3] and [1.4].

We assume that the H function of [1.1] satisfies the five conditions above and that the H function of [1.4] satisfies these conditions with β_i replaced by μ_i and that H function of [1.4] satisfies these conditions with α_i replaced by λ_i and β_i replaced by μ_i , i = 1,2,……………,n. We also assume that a common value of σ_0 can be found for all H functions involved in these equations.

RESULTS USED IN THE PROOF OF THE SEQUEL

Mellin Transform: We make much use of the Mellin transform. We denote the Mellin transform of f(u) by $M\{f(u)\} = F(s)$. Formally we have

$$M\{f(u)\} = F(s) = \int_0^\infty f(u)u^{s-1}\,du \qquad [2.1]$$

Inverse Mellin Transform: If $M\{f(u)\} = F(s)$ we shall write $f(u) = M^{-1}\{F(s)\}$ and inverse Mellin transform is defined as

$$M^{-1}\{F(s)\} = f(u) = \frac{1}{2\pi i}\int_c F(s)\,x^{-s}ds \qquad [2.2]$$

where the contour C, in [2.2] is usually a straight line parallel to the imaginary axis in the complex s(= $\sigma + it$) plane, with equation $\sigma = \sigma_0$.

Parseval's Theorem for Mellin Transform:

If M {a (u)} =A(s) and M {f (u)} =F(s)

Then

$$\int_0^\infty a(u)\;f(u)du = \frac{1}{2\Pi i}\int_c A(s)F(1-s)ds \qquad [2.3]$$

Again the contour C is some straight line whose equation is of the form $\sigma = \sigma_0$.

From [2.1] or [2.2] it is easy to deduce that if f(ux) is considered to be a function of u with x as a parameter, where x > 0, then

$$M\{f(ux)\} = x^{-s}M\{f(u)\} = x^{-s}F(s) \qquad [2.4]$$

From [2.4] and [2.3] we may deduce that

$$\int_0^\infty a(ux)f(u)du = \frac{1}{2\pi i}\int_c A(s)x^{-s}F(1-s)ds \qquad [2.5]$$

and this is the form in which we shall use the Parseval theorem here.

Conditions for the validity of [2.1], [2.2] and [2.3] can be found in [10.§§ 1.29, 3.17 and 4.14].

From [1.0] and [2.2] we may infer that

$$M\left[H\left(u\left|\begin{matrix}\alpha_j, & a_j \\ \beta_j, & a_j\end{matrix} : n\right.\right)\right] = \prod_{j=1}^{n}\left\{\frac{\Gamma(\alpha_j + sa_j)}{\Gamma(\beta_j - sa_j)}\right\} \qquad [2.6]$$

and this follows from [1.0] if condition(vb) of §1 holds [10, Theorem 29, p. 46].

It may still be true, however, if only condition (va) holds

Fox's Beta Formulae:

Fox defined Beta formulae by following fractional integrals:

$$\int_0^x (x^{1/c} - \upsilon^{1/c})^{d-e-1}\, \upsilon^{\frac{e}{c}-s-1} d\upsilon = \frac{c\Gamma(d-e)\Gamma(e-cs)}{\Gamma(d-cs)} x^{\frac{d}{c}-\frac{1}{c}-s} \qquad [2.7]$$

Provided d>e and $\frac{e}{c}>\sigma$ where s= σ+it and 0 < x < 1

$$\int_x^\infty (\upsilon^{1/c} - x^{1/c})^{d-e-1}\, \upsilon^{\frac{1}{c}-\frac{d}{c}-s-1} d\upsilon - \frac{c\Gamma(d-e)\Gamma(e+cs)}{\Gamma(d+cs)} x^{-\frac{e}{c}-s} \qquad [2.8]$$

Provide d>e and $\frac{e}{c}>\sigma$ where s= σ+it and x >1

Fractional Erdelyi-Kober Operator:

Fox used the following generalized Erdelyi-Kober operators:

$$T[\gamma, \epsilon : m]\{f(x)\} = \frac{m}{\Gamma\gamma} x^{-\gamma m - c + m - 1} \int_0^x (x^m - \upsilon^m)^{\gamma-1}\, \upsilon^{e} f(\upsilon) d\upsilon \qquad [2.9]$$

Where 0< x < 1

$$R[\gamma, \epsilon : m]\,\{f(x)\} = \frac{m}{\Gamma\gamma} x^{\epsilon} \int_x^\infty (\upsilon^m - x^m)^{\gamma-1}\, \upsilon^{-\epsilon-\gamma m + m - 1} f(\upsilon) d\upsilon \qquad [2.10]$$

Where x > 1

The operator T exists. If f(x)$\in L_p(0,\infty)$, p>1 , γ>0 and ϵ> $(1-p)/p$ and If , f(x) can be differentiated sufficient number of times then the operator T exists for both negative and positive value of γ. The operator R exists. If f(x)$\in L_p(0,\infty)$, p$\geq$1 and If, f(x) can be differentiated sufficient number of times then the operator R exists. If m > ϵ > $-1/p$ while γ can take any negative or positive value.

3. Solution of quadruple integral equations: On using M $\{f(u)\}$ = F(s) we may apply the Parseval theorem [2.5] to integral equations [1.1] [1.2], [1.3] and [1.4] and from [2.6], rewrite these equations in the form

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\beta_i-sa_i)}\right\}x^{-s}F(1-s)\,ds = \psi_1(x),\ x \in (0,a) \qquad [3.1]$$

$$\frac{1}{\pi i}\int_{2c} \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\}x^{-s}F(1-s)\,ds = \psi_2(x),\ x \in (a.b) \qquad [3.2]$$

[3.2]

$$\frac{1}{\pi i}\int_{2c} \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\}x^{-s}F(1-s)\,ds = \psi_3(x),\ x \in (b,c) \qquad [3.3]$$

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\lambda_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\}x^{-s}F(1-s)\,ds = \psi_4(x),\ x \in (c,\infty) \qquad [3.4]$$

We assume that the contour C is the same straight line $\sigma = \sigma_0$ for all [3.1], [3.2] , [3.3] and [3.4].
[3.1], [3.2] , [3.3] and [3.4] can be deduced from [1.1], [1.2] , [1.3] and [1.4] only if we knowSomething about the properties of f(x) and since these are not known at presentwe must proceed formally. The method used here works more easily with [3.1],[3.2] , [3.3] and [3.4] than with [1.1], [1.2] , [1.3] and [1.4].

Now in integral equation [3.1] replacing x by v and multiplying both sides of equation [3.1] by $\left(x^{b_n} - v^{b_n}\right)^{\mu_n - \beta_n - 1} . v^{b_n\beta_n - 1}$ where $b_n = 1/a_n$ and integrating both sides of integral equations [3.1] with respect to v from 0 to x where x Є (0,ξ) and applying well known Fox's Beta formula [2.7] in equation [3.1], under the conditions of convergence $\mu_n > \beta_n$ and $b_n\,\beta_n = \beta_n/a_n > \sigma_0$ ($s = \sigma_0 + it$ on the line $\sigma = \sigma_0$). When fractional integration is introduced the first of these conditions may no longer be necessary and the second may be relaxed, then we get

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n-1}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\beta_i-sa_i)}\right\}\frac{\Gamma(\alpha_n+sa_n)}{\Gamma(\mu_n-sa_n)}\ x^{-s}\ F(1-s)\,ds$$

$$= \frac{b_n}{\Gamma(\mu_n-\beta_n)}\,x^{-b_n\mu_n + b_n}\int_0^x \text{QUOTE}\ \psi_1(v)\,dv$$

Where $0 < x < 1$ and $b_n = 1/a_n$. [3.5]

We use the Erdelyi- Kober operator T from [2.9] in equation [3.5] , for brevity we

Write $T\left[\mu_i - \beta_i,\ b_i\beta_i - 1 : b_i\right]\{\psi_1(x)\} = T_i\{\psi_1(x)\}$,

$b_i = {}^1/_{a_i}$, i= 1,2,3,…………………..,n. and $x \in (0,a)$ [3.6]

Then $T\left[\mu_n - \beta_n,\ b_n\beta_n - 1 : b_n\right]\{\psi_1(x)\} = T_n\{\psi_1(x)\}$, $x \in (0,a)$ [3.7]

Hence from [3.7] , the integral equation [3.5] can be written as

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n-1}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\beta_i-sa_i)}\right\}\frac{\Gamma(\alpha_n+sa_n)}{\Gamma(\mu_n-sa_n)}\, x^{-s}\, F(1-s)\, ds = T_n\{\psi_1(x)\},\ x \in (0,a) \quad [3.8]$$

Now repeating the same process in integral equation [3.8] for i= n-1, n-2,……..,3,2,1.

Then the equation [3.8] takes the form

$$\frac{1}{2\pi i}\int_c \prod_{i-1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\} x^{-s} F(1-s)\, ds = \prod_{i=1}^{n} T_i\{\psi_1(x)\}\ ;\ x \in (0,a) \quad [3.9]$$

Now again in integral equation [3.4] replacing x by v and multiplying both

sides of the equation [3.4] by $\left(v^{b_n} - x^{b_n}\right)^{\lambda_n - \alpha_n - 1}.\, v^{b_n - b_n\lambda_n - 1}$

where $b_n = {}^1/_{a_n}$ as before and integrating both sides of integral equation [3.4] with

respect to v from x to ∞ where $x \in (c, \infty)$ and applying well known Fox's Beta formula [2.8] in equation [3.4] under the condition of convergence $\lambda_n > \alpha_n$ at the lower limit and a $n^{\upsilon}0 + \alpha_n > 0$ at the upper limit. But when the fractional integration operator R is introduced some of these conditions may no longer be necessary.
Hence , as before , σ_0 is the real part of s and also x > 0, then we get,

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n-1}\left\{\frac{\Gamma(\lambda_i+sa_i)}{\Gamma(\mu_i\ \ sa_i)}\right\}\frac{\Gamma(\alpha_n+sa_n)}{\Gamma(\mu_n-sa_n)}\, x^{-s}\, F(1-s)\, ds$$

$$= \frac{b_n}{\Gamma(\lambda_n-\alpha_n)}\, x^{b_n\alpha_n}\int_x^{\infty}\left(v^{b_n} - x^{b_n}\right)^{\lambda_n-\alpha_n-1}.\, v^{b_n - b_n\lambda_n - 1}\,\psi_4(v)\, dv$$

[3.10]

together with x > 1 and $b_n = {}^1/_{a_n}$.

Now using the Erdelyi- Kober operator R from [2.10] in the equation [3.10] for brevity

We write $R\left[\lambda_i - \alpha_i\,,\ b_i\alpha_i : b_i\right]\{\psi_4(x)\} = R_i\{\psi_4(x)\}$, $x \in (c, \infty)$. [3.11]

together with $b_i = {}^1/_{a_i}$, $i = 1,2,3,\ldots\ldots\ldots\ldots, n.$.

Then R $\left[\lambda_n - \alpha_n,\ b_n\alpha_n : b_n\right]$ $\{\psi_4$ (x)} = R$_n\{\psi_4$ (x)} , x Є (c , ∞) [3.12]

together with b$_n$ = $1/a_n$.

Hence from [3.12] the integral equation [3.9] can be written as

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n-1}\left\{\frac{\Gamma(\lambda_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\}\frac{\Gamma(\alpha_n+sa_n)}{\Gamma(\mu_n-sa_n)}\, x^{-s} F(1-s)\, ds = R_n\{\psi_4(x)\}, x \in (c, \infty) \quad [3.13]$$

together with b$_n$ = $1/a_n$.

Now repeating the same process in integral equation [3.13] for i= n-1,n-2………………
……..,3,2,1.then the equation [3.13] takes the form

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\} x^{-s} F(1-s)\, ds = = \prod_{i=1}^{n} R_i\,\{\psi_4(x)\}, x \in (c, \infty) \quad [3.14]$$

Now the integral equations [3.1], [3.2] , [3.3] and [3.4] are transformed into three other integral equations [3.9], [3.2] , [3.3] and [3.14] having common kernel.

Now we set

$$p(x) = \begin{bmatrix} \text{□ QUOTE □ □□□ } \{\psi_1(x)\};\ x \in (0, a\text{□ QUOTE □ □ □}) \\ \psi_2(x), \qquad x \subset (a, b) \\ \psi_3(x), \qquad x \in (b, c) \\ \text{□ QUOTE □ □□□ } \{\psi_4(x)\}, x \in (c, \infty\text{□ QUOTE □ □ □}) \end{bmatrix}$$

[3.15]

Then integral equations [3.9], [3.2], [3.3] and [3.14] can be put in to the compact form as

$$\frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i+sa_i)}{\Gamma(\mu_i-sa_i)}\right\} x^{-s} F(1-s)\, ds = p(x) \quad [3.16]$$

In order to solve [3.16] for f(x) = M$^{-1}[F(s)]$, using the terminology of § 2.

For this purpose we use the generalized Fourier transform which consists of the reciprocity

$$\phi(x) = \int_0^\infty p(ux) f(u)\, du \quad [3.17]$$

$$f(x) = \int_0^\infty q(ux)\, \phi(u)\, du \quad [3.18]$$

The functions p(x) and q(x) are known as the kernels and the transformation is said to be symmetrical if p(x) = q(x) and unsymmetrical otherwise. A full and detailed account of this transform is given in [10, Chapter 8, especially § 8.9].

Not every pair of functions can form the kernels of a transform such as [3.16], [3.17]. With the Mellin transform notation of §2 let M$[p(u)]$ = P(s) and M$[q(u)]$ = Q(s). Then among the conditions required for the validity of [3.17], [3.18] is the satisfaction of the functional equation

P(s) Q(1-s) = 1 [3.19]

In addition to this P(s) and Q(s) must be bounded and regular in certain

Strips of the complex s-plane to the imaginary axis and also f(x) or $\phi(x)$ must belong to L $_{p}(0,\infty)$ for some p ≥ 1. Since our method is formal we need consider [3.19] only here.

We apply the Parseval theorem, [2.5] of §2 to the right – hand sides of [3.17] and [3.18]. Writing M$[f(u)]$ = F(s),

From [3.17] we obtain

$$\phi(x) = \frac{1}{2\pi i}\int_c P(s)\, x^{-s} F(1-s)\, ds \qquad [3.20]$$

Also , on writing M$[\phi(u)]$ = $\boldsymbol{\phi}(s)$ and using [3.19] we deduce from [3.18] that

$$f(x) = \frac{1}{2\pi i}\int_c \frac{1}{P(1-s)}\, x^{-s}\, \boldsymbol{\phi}(1-s)\, ds. \qquad [3.21]$$

Hence, if P(s) and $\phi(x)$ are known in [3.20] we can solve for f(x) = M$^{-1}[F(s)]$ by means of equation [3.21].

On applying this idea to [3.16] and writing M$[p(x)]$ = P(s) we deduce that

$$f(x) = \frac{1}{2\pi i}\int_c \prod_{i=1}^{n}\left\{\frac{\Gamma(\mu_i - a_i + s a_i)}{\Gamma(\alpha_i + a_i - s a_i)}\right\} x^{-s} P(1-s)\, ds \qquad [3.22]$$

This is the formal solution of [3.9], [3.2], [3.3] and [3.14] and many important properties of f(x) can be deduced from it. But, by using the Parseval theorem [2.5] of §2, we can transform the integral of [3.22] so that the equation takes the form

$$f(x) = \int_0^\infty H\left(ux \left| \begin{matrix} \mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i \end{matrix} : n \right.\right) p(u)\, du \qquad [3.23]$$

where p(x) is given by [3.15]. From conditions (iv) and (va) of § 1 the H function of [3.23] exists if we can find a constant σ_0 such that

$\sigma_0 a_i > a_i - \mu_i$, i = 1,2,…………,n, and $2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n}(2a_i + \alpha_i - \mu_i)$.

The solution [3.23] when written out in full becomes

$$f(x) = \int_0^a H\left(ux\left|\begin{matrix}\mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i\end{matrix} : n\right.\right) \Pi_{i=1}^{n} T_i\{\psi_1(u)\} du + \int_a^b H\left(ux\left|\begin{matrix}\mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i\end{matrix} : n\right.\right) \psi_2(u) du + \int_b^c H\left(ux\left|\begin{matrix}\mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i\end{matrix} : n\right.\right) \psi_3(u) du + \int_c^\infty H\left(ux\left|\begin{matrix}\mu_i & a_i, & a_i \\ \alpha_i + & a_i, & a_i\end{matrix} : n\right.\right) \Pi_{i=1}^{n} R_i\{\psi_4(u)\} du \qquad [3.24]$$

If in equations [1.1], [1.2] [1.3] and [1.4] a is replaced by 1 and c is replaced by 1 and in equation [1.1] on right hand side the function ψ_1 (x) is replaced by the function g(x) and in equation [1.4] on right hand side the function ψ_4 (x) is replaced by the function h(x) and we take a = b = c = 1 then the quadruple integral equations [1.1], [1.2], [1.3] and [1.4] are reduced to the dual integral equations [1.5] and [1.6] earlier considered by Fox[5, equations (6), (7), pp. 390] and the solution of

$$\int_0^\infty H\left(ux\left|\begin{matrix}\alpha_i & a_i \\ \beta_i & a_i\end{matrix} : n\right.\right) f(u)\, du = g(x), \quad 0 < x < 1 \qquad [1.5]$$

$$\int_0^\infty H\left(ux\left|\begin{matrix}\lambda_i & a_i \\ \mu_i & a_i\end{matrix} : n\right.\right) f(u)\, du = h(x), \qquad x > 1 \qquad [1.6]$$

is

$$f(x) = \int_0^1 H\left(ux\left|\begin{matrix}\mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i\end{matrix} : n\right.\right) \prod_{i=1}^{n} T_i\{g(u)\}\, du$$

$$+ \int_1^\infty H\left(ux\left|\begin{matrix}\mu_i - a_i, & a_i \\ \alpha_i + a_i, & a_i\end{matrix} : n\right.\right) \prod_{i=1}^{n} R_i\{h(u)\}\, du \qquad [3.25]$$

Fox has given the relation between the H function of order 1 and Bessel function [5, Relation (5), pp. 390] as follows

$$H\left(x\left|\begin{matrix}\alpha, & \frac{1}{2} \\ \beta, & \frac{1}{2}\end{matrix} : 1\right.\right) - 2x^{\alpha - \beta + 1} J_{\alpha + \beta - 1}(2x),$$

Where J denotes a Bessel function.

Fox[5,reduced (6) and (7) by using relation (5) when n=1,pp. 390 to (1) and (2) as a very special case,pp.389]

In the case when n=1 we can compare our result with known solutions of (1), (2).

A solution of (1), (2) for the case β = 0 is given by Peters,[9, equations(3.1), (3.2),pp.7] and the solution is given by [9, (3.8), pp.10].

On writing n=1, a_1= ½, α_1= $(\mu+\omega)$/2, β_1= $(\mu-\ \omega+2)/2$, λ_1=ν/2 and μ_1 = $(\nu+2)/2$ in [3.24] we obtain a formula which agrees completely with Peter's (3.8) of [9]. Our method is formal, however, and so does not give conditions for the validity of the solution.

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